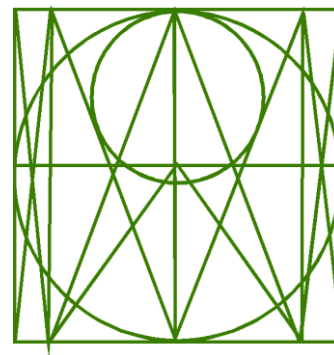

RATIONAL FUNCTIONS

Remember what we call a number like $\frac{2}{7}$? This is called a *rational number* because it is the *ratio* of two integers. In a like manner, a *rational function* is the ratio of two special functions called **polynomial functions**. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain of a such a function may not be all real numbers.



□ POLYNOMIAL FUNCTIONS

Each of the following is a polynomial function:

$y = 7$	(a <i>linear</i> function – it's a horizontal line)
$y = -3x + \sqrt{2}$	(a <i>linear</i> function – it's a line with slope = -3)
$y = 2x^2 - x + 9$	(a <i>quadratic</i> function – it's a parabola)
$f(t) = \sqrt[4]{2}t^3 - t^2$	(a <i>cubic</i> function)
$P(w) = -\pi w^4 + 5w^2 + 8$	(a <i>quartic</i> function)
$Q(a) = \frac{2}{3}a^5 - 4a + 1$	(a <i>quintic</i> function)

The key to any **polynomial function** is that all the exponents on the input variable come from the set of whole numbers: $\{0, 1, 2, 3, \dots\}$. The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in $\mathbb{R}$, the set of real numbers.

Consider the quartic (4th degree) polynomial function

$$y = -2\pi x^4 - \frac{9}{10}x^3 + 9.2x + \sqrt{2}$$

First look at the exponents; they are all whole numbers. Notice that the x in the term $9.2x$ has an understood exponent of 1. Even the last term, $\sqrt{2}$, can be written as $\sqrt{2}x^0$, and so even the exponent on this last term is a whole number.

Thus, all the exponents on the x 's (4, 3, 1, and 0) come from the whole numbers, while all the coefficients (-2π , $-\frac{9}{10}$, 9.2 , $\sqrt{2}$) come from $\mathbb{R}$. Considering the definition of polynomial function, the given function is indeed a polynomial function.

Each of the following is not a polynomial function:

$$y = \frac{1}{x} \quad \left(\frac{1}{x} = x^{-1} \text{ and } -1 \text{ is not a whole number}\right)$$

$$y = \sqrt{x} \quad \left(\sqrt{x} = x^{1/2} \text{ and } \frac{1}{2} \text{ is not a whole number}\right)$$

$$f(x) = \frac{1}{\sqrt[3]{x}} \quad \left(\frac{1}{\sqrt[3]{x}} = x^{-1/3} \text{ and } -\frac{1}{3} \text{ is not a whole number}\right)$$

$$g(x) = |x - 1| \quad (\text{no absolute values allowed around the } x)$$

$$E(x) = 2^x \quad (\text{since } x \text{ is in the exponent, it can be any number})$$

$$T(x) = \sin x \quad (\text{it's on your calculator, but it's not a polynomial function})$$

$$y = \log x \quad (\text{a log function can never be a polynomial function})$$

$$x^2 + y^2 = 25 \quad (\text{it's a circle – it's not a function of any kind})$$

Homework

1. Explain why $y = \pi x^5 - \sqrt{2}x^3 + \frac{1}{4}x - 17.5$ is a polynomial function.

2. Explain why $h(x) = \sqrt[3]{5}x^4 - \sqrt{2x} + \frac{1}{2}$ is not a polynomial function.
3. The highest exponent on the variable in a polynomial function is called its **degree**. Find the degree of each polynomial function:
 - a. $y = \pi$
 - b. $y = x^3 - 17x^2 + 8$
 - c. $y = \sqrt{2}x^8 - \frac{\pi}{2}x^{10}$
 - d. $y = 7x + 5$
4.
 - a. What is the domain of any polynomial function?
 - b. T/F: Every parabola is a polynomial function.
 - c. T/F: Every non-vertical line is a polynomial function.
 - d. T/F: Every line is a polynomial function.

□ RATIONAL FUNCTIONS

A **rational function** is defined to be the *ratio* of two *polynomial functions*. If P and Q are polynomial functions, then $R = \frac{P}{Q}$ is a

rational function. A typical example of a rational function is

$$y = \frac{3.9x^2 + 7x - 9}{x + 8} \text{ (a quadratic function over a linear function).}$$

EXAMPLE 1: **Graph:** $y = \frac{1}{x-2}$

Solution: Since y is the ratio of a constant polynomial function and a linear polynomial function, we know that y is a rational function.

We begin our analysis of this rational function by determining the **domain**. In order that the fraction be defined, we must not divide by zero. What value of x makes the denominator zero? The value $x = 2$ will. (Just set $x - 2 = 0$ and solve for x .)

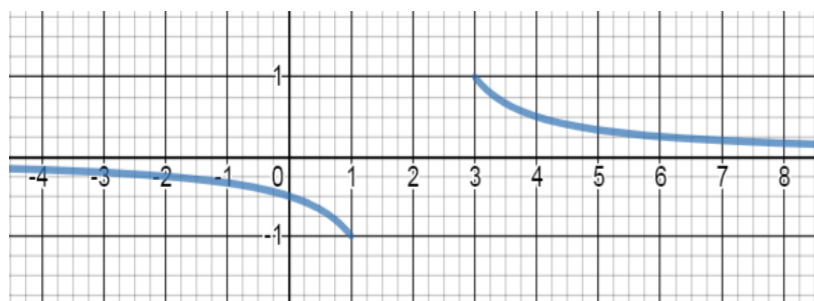
Therefore, the domain is all real numbers except 2; that is, the domain is $\mathbb{R} - \{2\}$.

Intercepts come next. If $x = 0$, then $y = \frac{1}{0-2} = -\frac{1}{2}$. Thus, $(0, -\frac{1}{2})$ is the y -intercept. To find an x intercept, set $y = 0$. This gives $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$, which has no solution. Thus, there are **no** x -intercepts.

Now for some ordered pairs that satisfy the formula $y = \frac{1}{x-2}$:

x	y
-3	$-\frac{1}{5}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	-1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$

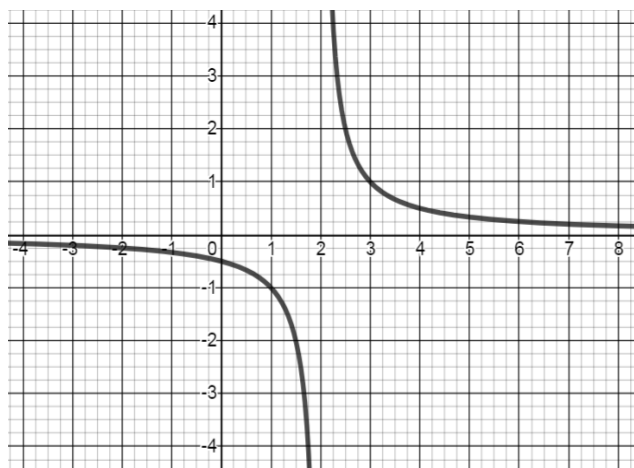
If we plot these points and connect them with a smooth curve, we would get the following graph:



What some students do at this point is to lazily connect the points $(3, 1)$ and $(1, -1)$ with a straight line. Talk about jumping to conclusions! Our domain of $\mathbb{R} - \{2\}$ implies that x cannot be 2 in this function; the straight-line trick won't work. So we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some x -values that are near 2:

$$\begin{array}{lll} (1\frac{1}{2}, -2) & (1\frac{3}{4}, -4) & (1\frac{7}{8}, -8) \\ (2\frac{1}{2}, 2) & (2\frac{1}{4}, 4) & (2\frac{1}{8}, 8) \end{array}$$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some real cool **limits**. Suppose we let x approach ∞ . The y -values are positive (the curve is above the x -axis), but are getting smaller and smaller, approaching zero. Thus, **as $x \rightarrow \infty, y \rightarrow 0$** . [This can be read: “As x grows infinitely large, y is getting closer and closer to 0.”]

Now let x approach $-\infty$. The y -values are negative but are rising toward zero. Therefore, **as $x \rightarrow -\infty, y \rightarrow 0$** .

The number 2 seems to be an interesting x -value. Although x can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line $x = 2$. In fact, if we let x approach 2 from the right, the curve is growing taller and taller, and so we have the limit: **As $x \rightarrow 2$ (from the right), $y \rightarrow \infty$** . [This can be read: “As x gets closer and closer to 2, approaching 2 from the right (meaning values larger than 2), y is growing infinitely large.”]

Now let x approach 2 from the left. This time the curve is dropping rapidly, toward negative infinity. This observation yields the limit: **As $x \rightarrow 2$ (from the left), $y \rightarrow -\infty$** .

Let's summarize the four limits we've deduced:

- ♦ As $x \rightarrow \infty$, $y \rightarrow 0$. $\lim_{x \rightarrow \infty} y = 0$
- ♦ As $x \rightarrow -\infty$, $y \rightarrow 0$. $\lim_{x \rightarrow -\infty} y = 0$
- ♦ As $x \rightarrow 2$ (from the right), $y \rightarrow \infty$. $\lim_{x \rightarrow 2^+} y = \infty$
- ♦ As $x \rightarrow 2$ (from the left), $y \rightarrow -\infty$. $\lim_{x \rightarrow 2^-} y = -\infty$

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the x -axis? We say that the line $y = 0$ (which is the x -axis) is a ***horizontal asymptote***.

Now look at the region of the graph near $x = 2$. The curve gets closer and closer to the vertical line $x = 2$ (in fact, on both sides of the vertical line). We call the line $x = 2$ a ***vertical asymptote***.

EXAMPLE 2: **Graph:** $y = \frac{2x-1}{x+2}$

Solution: First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when $x = -2$. Thus, the domain is $\mathbb{R} - \{-2\}$.

Now let's explore the **intercepts**:

If $x = 0$, then $y = \frac{2(0)-1}{0+2} = -\frac{1}{2}$. There's a y -intercept at $(0, -\frac{1}{2})$.

If $y = 0$, then $0 = \frac{2x-1}{x+2} \Rightarrow 2x-1 = 0 \Rightarrow x = \frac{1}{2}$. So $(\frac{1}{2}, 0)$ is an x -intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

$(-1, -3)$ $(1, 0.33)$ $(3, 1)$ $(5, 1.29)$ $(10, 1.58)$

(15, 1.71) (20, 1.77) (100, 1.95) (1000, 1.995)

What's happening as x grows very large? It appears that y is approaching 2. That is, **as $x \rightarrow \infty$, $y \rightarrow 2$** .

Now we'll let x go the other direction:

(-3, 7) (-5, 3.67) (-10, 2.63) (-20, 2.28)
(-100, 2.05) (-1000, 2.01)

These points show that **as $x \rightarrow -\infty$, $y \rightarrow 2$** .

Finally, here are some ordered pairs for x 's near -2 (the only real number not in the domain):

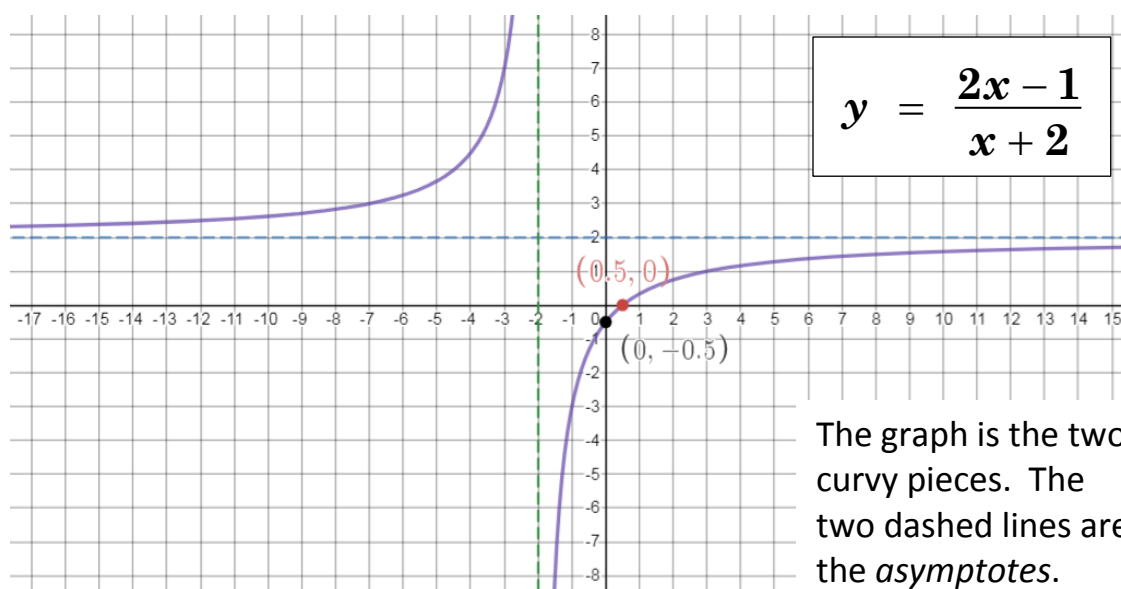
(-1.5, -8) (-1.9, -48) (-1.99, -498)

Thus, **as $x \rightarrow -2$ (from the right), $y \rightarrow -\infty$** .

(-2.5, 12) (-2.1, 52) (-2.01, 502)

So, as $x \rightarrow -2$ (from the left), $y \rightarrow \infty$.

Plotting as many of the calculated points as possible, including the two intercepts, and considering the four limits we've found, the following graph emerges:



We can now be reasonably sure of the **asymptotes** (denoted by the dashed lines). Either by recalling the limits described above or by looking at the graph, we conclude that there's a **vertical asymptote at $x = -2$** and a **horizontal asymptote at $y = 2$** .

EXAMPLE 3: **Graph:** $y = \frac{4}{1+x^2}$

Solution: Why is this function rational? Because it's the *ratio* $\frac{P}{Q}$ of two polynomial functions: the constant polynomial

$P(x) = 4$ and the quadratic polynomial $Q(x) = 1 + x^2$.

To find the **domain**, set the denominator to zero to see what's not in the domain: $1 + x^2 = 0$. This equation has no solution in $\mathbb{R}$, since solving it leads to $x = \pm\sqrt{-1}$, which are not real numbers. In fact, for any value of x , the quantity $1 + x^2$ is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is $\mathbb{R}$. We can also figure that the graph will not have a **vertical asymptote**, since the denominator can never be 0.

Notice that if we put in some positive value of x , we'll get a certain y -value. Now look at what will happen if we put $-x$ (the *opposite* of x) into the formula. Since $(-x)^2$ is equal to x^2 , we will get the same y -value. This implies that the left side of the graph will be the mirror image of the right side. We say that the graph has ***y-axis symmetry*** (or is *symmetric with respect to the y-axis*).

Now we seek the **intercepts**. Set $x = 0$ to get $y = 4$, and so the y-intercept is $(0, 4)$. Now set $y = 0$, giving

$$0 = \frac{4}{1+x^2} \Rightarrow 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \Rightarrow 0 = 4$$

This absurd result indicates that the equation has no solution; hence, there are **no** x-intercepts.

It's time for some additional ordered pairs for this function:

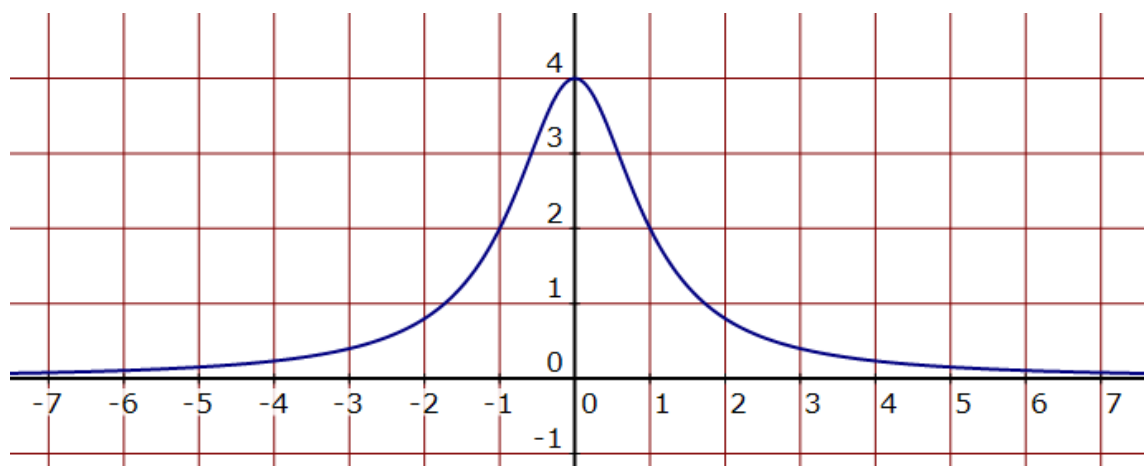
(1, 2) (2, 0.8) (3, 0.4) (4, 0.24) (10, 0.04) (200, 0.0001)

These points suggest the limit: **As $x \rightarrow \infty, y \rightarrow 0$.** This implies that **$y = 0$ is a horizontal asymptote.**

Here are some more ordered pairs, designed to see what happens as we approach the y-axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

If we plot all the points calculated so far, and if we recall the y-axis symmetry, we get the following graph:



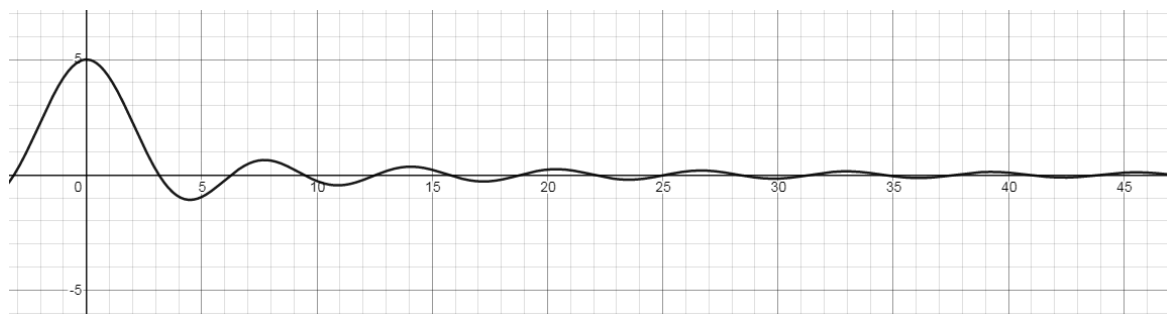
We determined at the outset that the *domain* of this rational function is $\mathbb{R}$. Is it clear from the graph that this is indeed the case?

If you happen to know what the **range** of a function means, you can see that the range of this function is all real numbers that are greater than 0 but less than or equal to 4: **Range = $(0, 4]$**

Homework

5. Consider the rational function in Example 2: $y = \frac{2x-1}{x+2}$.
Without referring to the graph, prove that y can have the value 2.01, but y can never have the value 2.
6. Find the domain:
- a. $y = \frac{2x+7}{9}$ b. $f(x) = \frac{2x-3}{4+x}$
- c. $g(x) = \frac{3x}{2x-10}$ d. $R(x) = \frac{x-1}{-7x+4}$
- e. $f(x) = \frac{x^2-9}{x^2-100}$ f. $g(x) = \frac{8x-16}{x^2+25}$
7. Find the intercepts:
- a. $f(x) = \frac{x-4}{x-2}$ b. $y = \frac{3}{5x-15}$
- c. $y = \frac{2x+1}{x-3}$ d. $g(x) = \frac{5-x}{6x+1}$
8. Find the asymptotes:
- a. $R(x) = \frac{8x+1}{4x-4}$ b. $y = \frac{2x-3}{2x+1}$
- c. $y = \frac{3x-7}{x+2}$ d. $h(x) = \frac{2x+7}{4x-4}$
9. Find the domain, the intercepts, and the asymptotes of
$$y = \frac{1}{4+x^2}$$
10. Perform a complete analysis of the function $y = \frac{2}{x-3}$.

11. Perform a complete analysis of the function $y = \frac{3x-5}{x-2}$.
12. Perform a complete analysis of the function $y = \frac{2}{2+x^2}$.
13. Perform a complete analysis of the function $y = \frac{-1}{x+1}$.
14. Consider the graph



Explain why the horizontal line $y = 0$ (that is, the x -axis) is a horizontal asymptote for the curve.

Review Problems

15.
 - a. Explain why $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$ is a polynomial function. What is its degree?
 - b. Explain why $y = 3x^5 - \sqrt{x} + \pi$ is not a polynomial function.
16.
 - a. A horizontal line (is, is not) a polynomial function.
 - b. The function $y = \frac{1}{x}$ (is, is not) a polynomial function.
 - c. What is the degree of the polynomial function $y = 7x - \pi$?
 - d. Is a circle a polynomial function?

17. Consider the rational function $y = \frac{7}{2x-8}$.
- Find the domain.
 - Find all the intercepts.
 - Find all the asymptotes.
 - Calculate y if $x = 4.1$.
18. Find all the intercepts and asymptotes of $r(x) = \frac{8x+6}{2x-3}$, and graph.
19. Graph $y = \frac{-2x-2}{x-1}$.
- As $x \rightarrow 1$ (from the right), $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow 1$ (from the left), $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow \infty$, $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow -\infty$, $y \rightarrow \underline{\hspace{1cm}}$.
20. Graph $y = \frac{5}{2+x^2}$. Discuss domain, symmetry, and asymptotes.
- As $x \rightarrow \infty$, $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow -\infty$, $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow 0$ (from the right), $y \rightarrow \underline{\hspace{1cm}}$.
 As $x \rightarrow 0$ (from the left), $y \rightarrow \underline{\hspace{1cm}}$.
21. True/False:
- $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$ is a polynomial function.
 - $y = \frac{1}{x^5}$ is a polynomial function.
 - The graph of $f(x) = \frac{1}{2x+10}$ has a vertical asymptote at $x = -5$.
 - The graph of $g(x) = \frac{10x+9}{5x-11}$ has a horizontal asymptote at $y = 10$.
 - The domain of the function $y = \frac{6}{1+x^2}$ is $\mathbb{R} - \{\pm 1\}$.

- f. For the graph of $y = \frac{3x+1}{x-\pi}$, as $x \rightarrow \infty$, $y \rightarrow 3$.

Solutions

1. All coefficients are from $\mathbb{R}$, and all exponents are from $\mathbb{W}$ (the whole numbers).
2. The middle term is $\sqrt{2}x^{1/2}$, and $\frac{1}{2} \notin \mathbb{W}$.
3. a. 0 b. 3 c. 10 d. 1
4. a. $\mathbb{R}$ b. False c. True d. False
5. $y = \frac{2x-1}{x+2} \Rightarrow 2.01 = \frac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502$.
So, $(-502, 2.01)$ is on the graph, and indeed y can be 2.01.

Now let's pretend that y could be 2; then

$$2 = \frac{2x-1}{x+2} \Rightarrow 2x + 4 = 2x - 1 \Rightarrow 4 = -1 \Rightarrow \text{No solution. Thus,}$$

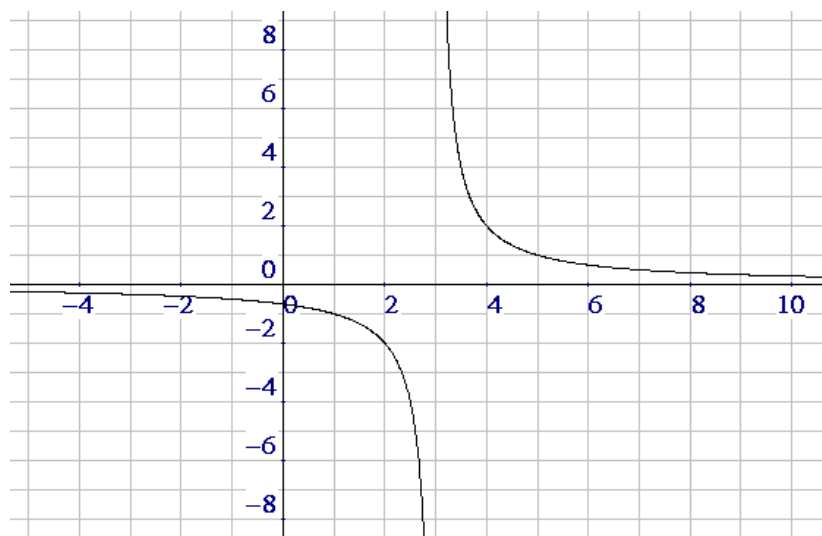
there is no x which will make $y = 2$.

6. a. $\mathbb{R}$ b. $\mathbb{R} - \{-4\}$ c. $\mathbb{R} - \{5\}$ d. $\mathbb{R} - \left\{\frac{4}{7}\right\}$ e. $\mathbb{R} - \{\pm 10\}$ f. $\mathbb{R}$
7. a. $(4, 0)$ $(0, 2)$ b. $(0, -\frac{1}{5})$ c. $(-\frac{1}{2}, 0)$ $(0, -\frac{1}{3})$ d. $(5, 0)$ $(0, 5)$
8. a. $x = 1$ $y = 2$ b. $x = -\frac{1}{2}$ $y = 1$ c. $x = -2$ $y = 3$ d. $x = 1$ $y = \frac{1}{2}$
9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is $\mathbb{R}$.

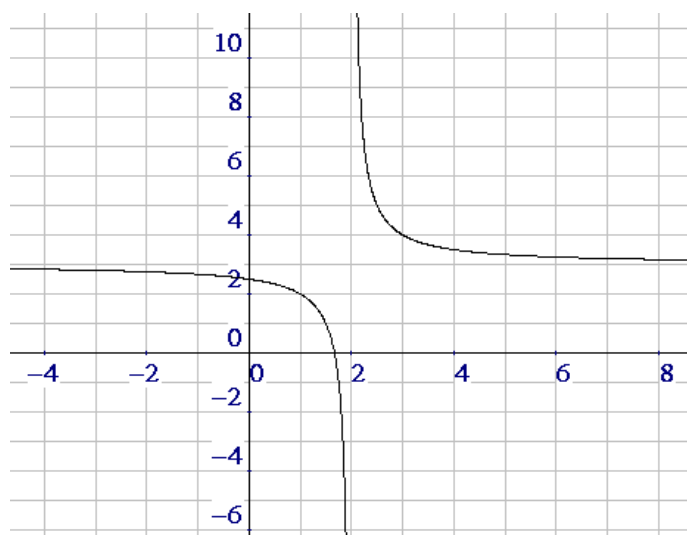
Setting $x = 0$ gives a y -value of $1/4$, so the y -intercept is $(0, \frac{1}{4})$. If you set $y = 0$, you'll get no solution for y . Thus, there is no x -intercept.

There are no vertical asymptotes, since the denominator is never zero. Letting x approach either ∞ or $-\infty$, y approaches 0. Thus, a horizontal asymptote is $y = 0$ (the x -axis).

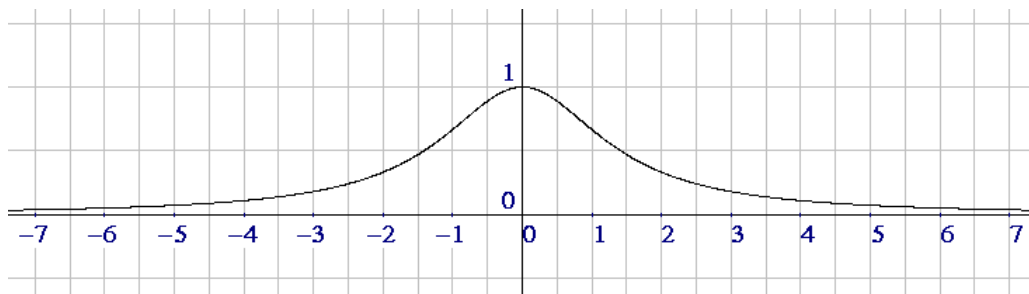
10.

Domain = $\mathbb{R} - \{3\}$ x -int: none y -int: $(0, -\frac{2}{3})$ As $x \rightarrow 3$ (from the right), $y \rightarrow \infty$ As $x \rightarrow 3$ (from the left), $y \rightarrow -\infty$ As $x \rightarrow \infty$, $y \rightarrow 0$ As $x \rightarrow -\infty$, $y \rightarrow 0$ vert asy: $x = 3$ horiz asy: $y = 0$

11.

Domain = $\mathbb{R} - \{2\}$ x -int: $(\frac{5}{3}, 0)$ y -int: $(0, \frac{5}{2})$ As $x \rightarrow 2$ (from the right), $y \rightarrow \infty$ As $x \rightarrow 2$ (from the left), $y \rightarrow -\infty$ As $x \rightarrow \infty$, $y \rightarrow 3$ As $x \rightarrow -\infty$, $y \rightarrow 3$ vert asy: $x = 2$ horiz asy: $y = 3$

12.

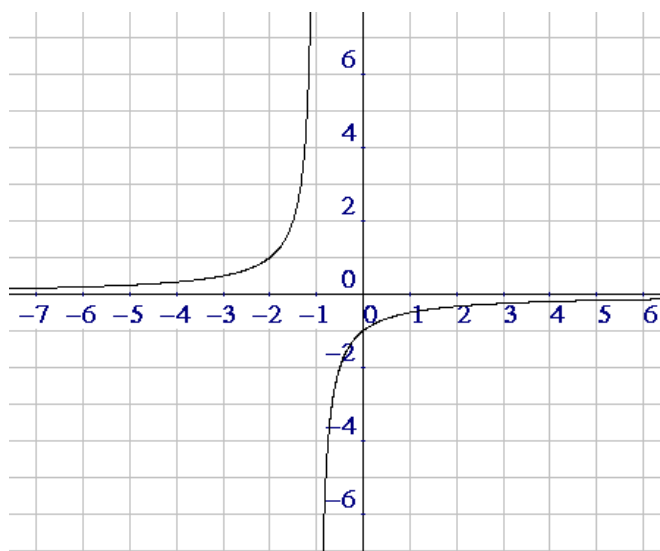
Domain = $\mathbb{R}$ x -int: none y -int: (0, 1)As $x \rightarrow \infty$, $y \rightarrow 0$ As $x \rightarrow -\infty$, $y \rightarrow 0$

vert asy: none

horiz asy: $y = 0$

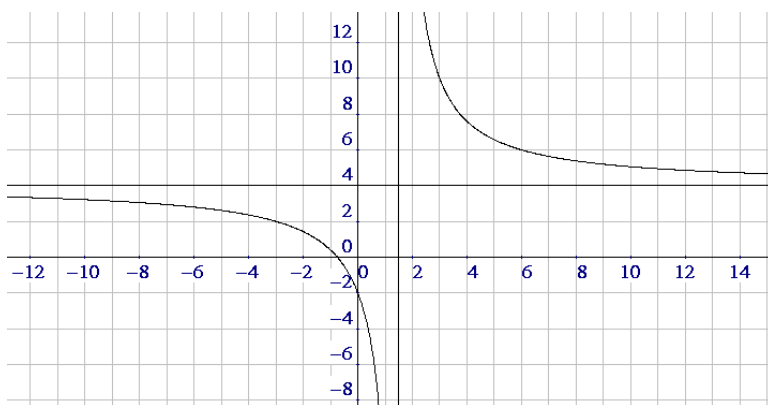
maximum point at (0, 1)

13.

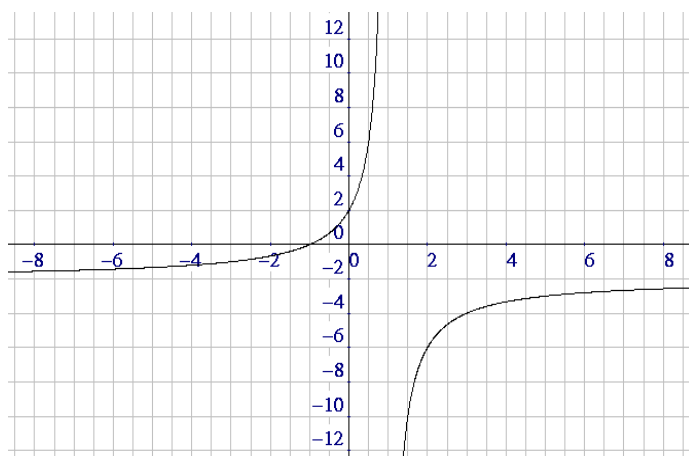
Domain = $\mathbb{R} - \{-1\}$ x -int: none y -int: (0, -1)As $x \rightarrow -1$ (from the right), $y \rightarrow \infty$ As $x \rightarrow -1$ (from the left), $y \rightarrow -\infty$ As $x \rightarrow \infty$, $y \rightarrow 0$ As $x \rightarrow -\infty$, $y \rightarrow 0$ vert asy: $x = -1$ horiz asy: $y = 0$

14. Because of the limit: As $x \rightarrow \infty$, $y \rightarrow 0$. Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the x -axis (the line $y = 0$), and this is ultimately what is meant by a horizontal asymptote.

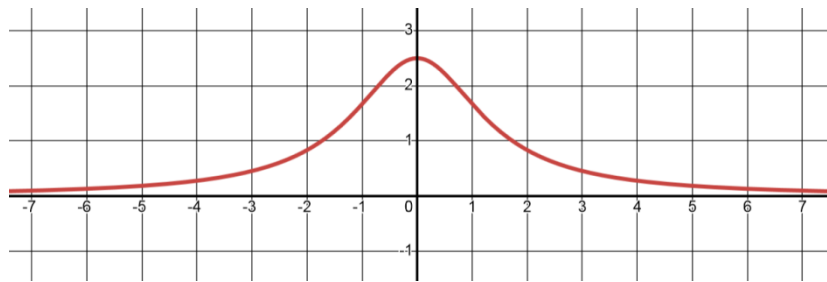
15. a. f is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.
 b. Look at the middle term; it can be written as $x^{1/2}$, a term whose exponent is not from the whole numbers.
16. a. is b. is not ($1/x = x^{-1}$) c. 1 d. It's not even a function, let alone the special function called a polynomial.
17. a. $\mathbb{R} - \{4\}$ b. $(0, -7/8)$ c. $x = 4$ and $y = 0$ d. 35
18. Intercepts: $(0, -2)$ and $(-\frac{3}{4}, 0)$; vert asy: $x = \frac{3}{2}$; horiz asy: $y = 4$



19.

Limits: $-\infty$; ∞ ; -2 ; -2

20.

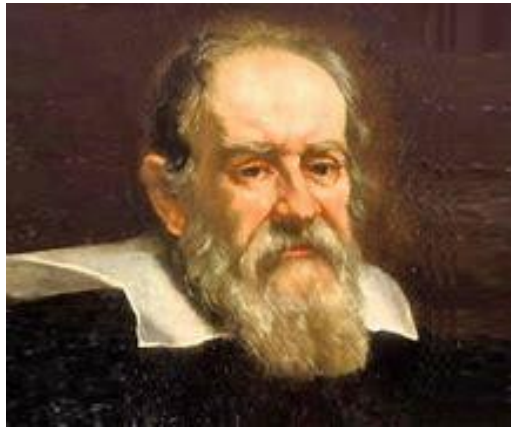
Domain = $\mathbb{R}$ y -axis symmetry

No vert asy

Horiz asy: $y = 0$ Limits: 0 ; 0 ; $\frac{5}{2}$; $\frac{5}{2}$

21. a. T b. F c. T d. F e. F f. T

“ The universe cannot be read until we have learned the language in which it is written. It is written in mathematical language.”



Galileo Galilei